Biased growth processes and the "rich-get-richer" principle

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We study a simple stochastic system with a "rich-get-richer" behavior, in which there are 2 states, and N particles that are successively assigned to one of the states, with a probability p_i that depends on the states' occupation n_i as $p_i = n_i^\gamma/(n_1^\gamma + n_2^\gamma)$. We show that there is a phase transition as γ crosses the critical value $\gamma_c = 1$. For $\gamma < 1$, in the thermodynamic limit the occupations are approximately the same, $n_1 \approx n_2$. For $\gamma > 1$, however, a spontaneous symmetry breaking occurs, and the system goes to a highly clustered configuration, in which one of the states has almost all the particles. These results also hold for any finite number of states (not only two). We show that this "rich-get-richer" principle governs the growth dynamics in a simple model of gravitational aggregation, and we argue that the same is true in all growth processes mediated by long-range forces like gravity.

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The first quantitative formulation of the "rich-get-richer" principle was in the context of economics, by Simon [1]. He proposed that businesses grow at an average rate that is proportional to their sizes. He developed a simple stochastic model incorporating this principle, which was able to explain the observed size distribution of real firms. The same law of a growth rate proportional to size also helps explain the distributions of city sizes [2,3], income ("Pareto's law"), word frequency in texts [2], and maybe even scientific impact ("Mathew's law") [4]. The "rich-get-richer" principle has attracted much interest recently, due to the discovery by Barabási and Albert [5,6] that it is responsible for the power-law degree distribution found in most networks, like the Internet [7], the World Wide Web [8], biological networks [9], language networks [10], and many others.

In this Letter, we are interested in understanding the statistical physics of systems described by skewed distributions of the "rich-get-richer" type. In the studies mentioned above, the systems are composed of many basic components (thousands of cities, or hundreds of thousands of words, etc.), whose number changes with time. We want to capture the essential features of "rich-get-richer" accumulation processes, without the complication of a system with a variable number of component (or "states"). With this purpose in mind, we study a very simple model, described by two states S_1 and S_2, and N "particles." The particles are assigned sequentially one by one to one of the two states according to a stochastic process, with assignment probabilities that favor the state with the greater current occupancy. Thus, our model is close to some classical urn models of probability theory [11]. We later tackle the case of a number m of states greater than 2, and we will see that all our conclusions remain the same as in the simpler case of $m=2$.

Another motivation for the present work is that previous work on "rich-get-richer" accumulation phenomena has almost entirely been focused on very complex systems, such as sociological and biological networks, the Internet, and so on. Although data on such systems are often abundant, it is hard to assess their underlying dynamical mechanisms from first principles. We will show in this paper that a rich-get-richer dynamics exists in a simple gravitational aggregation system. Thus, this kind of biased growth process also occurs in "hard" physical systems.

In our model, at every time unit a particle goes either into state S_1 with probability $p_1(n_1, n_2)$, or into state S_2 with probability $p_2(n_1, n_2)$, where n_1 and n_2 are the number of particles already present in states S_1 and S_2, respectively. This process is repeated until all N particles are assigned a state. This could represent, for instance, two cities competing for a total population of N persons. In this context, the likelihood of a given person going to S_1 or S_2 will in general depend on how many people already live in S_1 and S_2. In the case of gravitational aggregation, a particle can fall into one of two bodies (S_1 and S_2), with the corresponding probabilities depending on their relative mass. The simple "rich-get-richer" principle used by Simon and others corresponds in our model to taking p_i to be $n_i/(n_1+n_2)$, with $i=1,2$. We will assume here a more general dependence, of the form $p_i = n_i^\gamma/(n_1^\gamma + n_2^\gamma)$, where γ is a real parameter. $\gamma > 0$ corresponds to a generalized "rich-get-richer" law, since particles tend to go to the most occupied state. We think this general law for the probabilities is appropriate for several realistic systems, and in particular we will show that it is a good approximation of an idealized model of gravitational aggregation. This stochastic process bears some resemblance to some well-known models of nonequilibrium statistical physics [13], like the biased random walk [14], the Bragg-Williams kinetic model, and models described by run-time statistics [15], besides the urn models already mentioned [11]. We note that Krapivsky, Redner, and Leyvraz have used a generalized "rich-get-richer" law similar to the one we propose, in the context of complex networks [12]. Although their system is very different from the one we use in this paper, it is interesting to compare their results to ours, and we do so later in the paper. The case of more than 2 states is essentially similar, and will be treated in the end of the paper. Essentially all the interesting phenomena are the same as in the case of 2 states, and thus we will study this simple case in detail first.

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The most important quantity in our system is the probability distribution $P_N(n_1)$, which measures the probability that n_1 particles went to state S_1, and $n_2=N-n_1$ particles went to state S_2. Our main result is that, in the thermodynamic limit $N\to\infty$, there is a phase transition as γ crosses the critical value $\gamma_c=1$, in which the statistical properties of the system change dramatically. More precisely, for $\gamma<1$, the distribution $P_N(n_1)$ has a single narrow peak around the mean value $\bar{n}_1=N/2$. In other words, we can expect that for large N the number of particles found in one state is likely to be very close to the number of particles in the other state, both being close to the average $N/2$. In this case, we have $\delta n_1/\bar{n}_1 \ll 1$, where δn_1 is the size of the typical fluctuation around the average. In an aggregation process, this would mean that both bodies end up with about the same mass. If $\gamma>1$, on the other hand, we show that $P_N(n_1)$ has two narrow peaks, located around the extreme values $n_1=0$ and $n_1=N$. This means that for these values of γ, the bias in the growth process is so strong that almost all particles go to one of the two states, the other one remaining practically empty. The richer person gets almost everything, and the poor one gets nearly nothing. In the example of gravitational aggregation, for $\gamma>1$ one body ends up with almost all the mass, while the other one has negligible mass. The average $\bar{n}_1$ (and $\bar{n}_2$) is still equal to $N/2$, but in this case the average has little physical meaning, since there is practically no chance of getting n_1 or n_2 close to the average value in any given realization. Unlike the case of $\gamma<1$, the final outcome of the process is completely indeterminate in the statistical sense: as a result of the double-peaked nature of the distribution function P_N, the system will end up in one of two *macroscopically distinguishable* final states. In this case, $\delta n_1/\bar{n}_1$ is of order 1. We show the existence of this transition analytically through a mean-field approximation, and verify this prediction numerically. We explain this transition as a result of the competition between the tendency to cluster due to the bias in the growth process, and the fact that the number of ways to reach unclustered states (with n_1 close to n_2) is larger than to reach clustered states (with n_1 very different from n_2). We also show that for $\gamma>1$, a phenomenon similar to Bose-Einstein condensation occurs: in the limit $N\to\infty$, there is a nonzero probability that *all* particles go to one state, the other state being empty. To illustrate these results with a physical system, we introduce an idealized model for aggregation of matter by gravitation, and show that it is an example of a system with $\gamma>1$. We argue that this is a general feature of aggregation processes, if long-range forces are present.

We start by deriving the master equation for the biased stochastic process described above. Suppose that at some point in the process there are n_1 particles in state S_1, and n_2 particles in state S_2. Then the probability that the next particle will go to state S_i is $p_i(n_1,\ n_2), i=1,2$, where

$$p_i(n_1,n_2) = \frac{(n_i+1)^\gamma}{(n_1+1)^\gamma + (n_2+1)^\gamma}. \tag{1}$$

The reason we use $p_i \sim (n_i+1)^\gamma$ is that we avoid problems in the important case where initially $n_i=0$. For large n_i, Eq. (1) is approximately the same as $p_i=n_i^\gamma/(n_1^\gamma+n_2^\gamma)$. A positive γ means that there is an "attraction" between the particles, or a preference for new particles to go to the most occupied state, and thus a tendency to clustering. $\gamma<0$ would indicate a "repulsion" between particles. We focus on the $\gamma>0$ case in this paper.

We denote by (n_1,n_2) the configuration of the system where there are n_1 particles in state S_1, and n_2 particles in S_2. Let $P(n_1,n_2)$ be the probability that, after n_1+n_2 steps, state (n_1,n_2) is reached. This state can arise in one of two ways: either the system was in state (n_1-1,n_2) and then the next particle went to state S_1, or the system was in the state (n_1,n_2-1), and the next particle went to S_2. This allows us to write $P(n_1,n_2)$ as the sum of two terms corresponding to these two contributions, using Eq. (1):

$$\begin{aligned} P(n_1,n_2) &= \frac{n_1^\gamma}{n_1^\gamma + (n_2+1)^\gamma} P(n_1-1,n_2) \\ &\quad + \frac{n_2^\gamma}{(n_1+1)^\gamma + n_2^\gamma} P(n_1,n_2-1). \end{aligned} \tag{2}$$

We will assume that initially both states are empty. This corresponds to $P(0,0)=1$. Using this initial condition together with Eq. (2), we can calculate iteratively P for any value of n_1 and n_2.

In order to gain some insight on the behavior of the probability distribution P, we turn to a mean-field approach. Suppose we have a very large number N of particles, and that n_1 and n_2 are both much greater than 1. In this case, we can approximate n_1 and n_2 as continuous variables. The probabilistic rule given by Eq. (1) is hence well approximated on average by the continuous ordinary differential equations:

$$\dot{n}_1 = p_1(n_1,n_2), \quad \dot{n}_2 = p_2(n_1,n_2), \tag{3}$$

where we imagine that a large number of particles "fall" into either S_1 or S_2 at a constant rate, and the dot denotes derivative with respect to time. We are particularly interested in how the ratio $r=n_1/n_2$ changes with time. Using Eq. (3), we get

$$\dot{r} = \frac{d}{dt}\left(\frac{n_1}{n_2}\right) = \frac{\dot{n}_1}{n_2} - \frac{n_1\dot{n}_2}{n_2^2} = \frac{1}{n_2^2}\left(\frac{n_2 n_1^\gamma - n_1 n_2^\gamma}{n_1^\gamma + n_2^\gamma}\right), \tag{4}$$

where we used Eq. (1) and the fact that $n_1,n_2 \gg 1$.

From Eq. (4), we see that the rate of change of $r=n_1/n_2$ depends on the ratio $n_2 n_1^\gamma/n_1 n_2^\gamma = r^{\gamma-1}$,

$$\begin{aligned} &\text{if } r^{\gamma-1} > 1, \quad \dot{r} > 0, \\ &\text{if } r^{\gamma-1} < 1, \quad \dot{r} < 0. \end{aligned} \tag{5}$$

Let us consider first the case $\gamma<1$. In this case, the power $\gamma-1$ in Eq. (5) is a negative number. If there are more particles in state S_1 than in S_2, the ratio r is greater than 1, and therefore $r^{\gamma-1}$ is less than 1. From Eq. (5), this means that $\dot{r}<0$. Analogously, we find from Eq. (5) that if $r<1$, then $\dot{r}>1$. In other words, if $\gamma<1$, the dynamics given by Eq. (4) acts so as to keep the ratio $r=n_1/n_2$ close to 1, and thus n_1 close to n_2. Differences $\delta n=n_1-n_2$ between n_1 and n_2 are "dampened" in this case. These considerations suggest that for a large number N of particles, the system will be almost

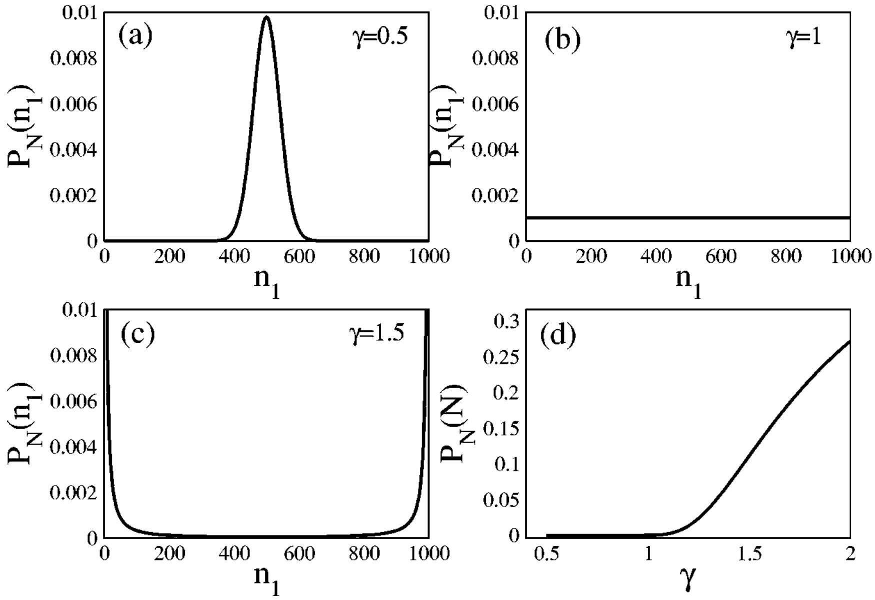

FIG. 1. Numerical calculation of $P_N(n_1)$ with $N=10^3$, for (a) $\gamma<1$, (b) $\gamma=1$, and (c) $\gamma>1$. There is a well-defined peak around $N/2$ for $\gamma<1$, and two peaks around 0 and N for $\gamma>1$, in accordance with the theory. (d) shows the probability $P_N(N)$ that all particles go to the state S_1, for $N=10^4$.

certainly in a state where $n_1 \approx n_2$, and if we look at (say) n_1, we will find a value very close to the average $N/2$. Even though the particular sequence in which the states S_1 and S_2 are filled is unpredictable, the "macroscopic" variables such as n_1 and n_2 are perfectly predictable, since they are overwhelmingly likely to be in a very narrow vicinity of their averages.

Next we consider the case $\gamma>1$. Now the power $\gamma-1$ is positive, and from Eq. (5), we have that $\dot{r}>0$ if $r>1$, and $\dot{r}<0$ if $r<1$. Here the situation is the opposite to the previous one: the mean-field dynamics pushes the ratio r away from 1, and small differences between n_1 and n_2 are amplified as more particles are successively assigned states. The mean-field result thus implies that in this case, the system will go to a highly clustered state, where the vast majority of particles are in one of the two states, the other one being nearly empty. Whether S_1 or S_2 will be the favored state is completely unpredictable, since this depends on random details early in the growth process. Accordingly, for $\gamma>1$ the system is *statistically unpredictable*: not only the details of the process are unpredictable, but also the final values of the macroscopic variables themselves cannot be predicted. The occupation probabilities (1) are symmetric with respect to the states S_1 and S_2, and therefore the average value of both n_1 and n_2 for many realizations is the same as in the $\gamma<1$ case: $\bar{n}_1=\bar{n}_2=N/2$. However, for any particular realization n_1 and n_2 will be far away from their averages, close to either 0 or N, and the relative standard deviation $\delta n_1/\bar{n}_1$ will be close to 1. For $\gamma>1$, a *spontaneous symmetry breaking* occurs: the final state of the system is one in which there is a preference for either S_1 or S_2, whereas the dynamics itself, given by Eq. (1), is symmetric. The transition at $\gamma=\gamma_c=1$ can be regarded as a first-order nonequilibrium phase transition.

For a given number N of particles, the quantity that characterizes the system is the probability distribution $P_N(n_1)$, which measures the probability that there are n_1 particles in S_1, given that there are $N=n_1+n_2$ particles in total. Because of the symmetry of Eq. (1), we could equally well use n_2 in the definition of P_N, the results would be exactly the same. P_N is given in terms of the previously defined distribution by $P_N(n_1)=P(n_1,N-n_1)$. The considerations of the previous paragraphs imply that in the limit of large N, for $\gamma<1$, $P_N(n_1)$ has a sharp peak around the average value $\bar{n}_1=N/2$, practically vanishing for values of n_1 (and n_2) far away from $N/2$. For $\gamma>1$, on the other hand, $P_N(n_1)$ should have two peaks, around the extreme values $n_1=0$ and $n_1=N$, corresponding to the preference in this case for clustered states. The case $\gamma=1$ is a threshold, and from the mean-field result (4), we expect that P_N will be a constant value, yielding a uniform occupation probability.

In order to verify the above predictions, we solve numerically, by successive iterations, Eq. (2), with the initial condition $P(0,0)=1$, corresponding to a state where no particles are initially assigned. The results are plotted in Figs. 1(a)–1(c) for the cases $\gamma<1$, $\gamma=1$, and $\gamma>1$ respectively. We see that our predictions are indeed confirmed, and there is a "clustering transition" as γ crosses the critical value $\gamma_c=1$. This phenomenon can be understood by realizing that in our stochastic growth process there is a competition between two effects. First, the bias in the assignment of states given by Eq. (1) means that (for $\gamma>0$) the particles tend to cluster, which tries to push the distribution P_N away from the mean $N/2$ and toward the extremes 0 and N. On the other hand, there are many more different sequences of assignments which correspond to states where $n_1 \approx n_2 \approx N/2$. In other words, the *entropy* of unclustered states is greater. This is clearly seen if we remember that the case $\gamma=0$, corresponding to no bias, yields just the binomial distribution for P_N, which is highly peaked around the average. For $\gamma<1$, the entropy factor wins against the clustering tendency, and P_N has a peak around the average, as seen in Fig. 1(a). If $\gamma>1$, the bias is stronger, and clustering wins, which leads to the double peak in P_N. For $\gamma=\gamma_c=1$, the two opposing effects exactly balance, and all states are equally likely, which is consistent with the constant distribution shown in Fig. 1(b). The results in Fig. 1 were also verified by a direct Monte Carlo simulation of the growth process.

From Eq. (1), we can calculate exactly the probability $P_N(N)=P_N(0)$ for all particles to go to a single state (say, S_1). From Eq. (1), this is given by

$$P_N(N) = \frac{1}{2} \cdot \frac{2^\gamma}{1+2^\gamma} \cdot \frac{3^\gamma}{1+3^\gamma} \cdots \frac{N^\gamma}{1+N^\gamma}. \tag{6}$$

Taking the logarithm of both sides, and doing some simple algebraic manipulations, we get

$$-\ln P_N(N) = \ln 2 + \ln(1+2^{-\gamma}) + \cdots + \ln(1+N^{-\gamma}). \tag{7}$$

Now we go to the thermodynamic limit $N\to\infty$. The sum on the right-hand side of Eq. (7) becomes an infinite series. For $\gamma \leqslant 0$, we see immediately that this series diverges, and the probability $P=\lim_{N\to\infty}P_N(N)$ for all particles to go to one of the states is 0. For $\gamma > 0$, the terms inside the logarithms of Eq. (7) decrease, and for a given $\gamma > 0$ there is a number m such that $\ln(1+k^{-\gamma})$ is well approximated by its first-order Taylor expansion, if $k \geqslant m$. Hence, we can approximate Eq. (7) for $N\to\infty$ by

$$-\ln P = C + \sum_{n=m}^{\infty} \frac{1}{n^\gamma}, \tag{8}$$

where $C=\ln 2+\cdots+\ln(1+(m-1)^{-\gamma})$ is a finite positive constant. The convergence properties of the sum in (8) are well known, and we get the result

$$\begin{aligned} &\text{if } \gamma \leqslant 1, \quad \text{the sum diverges, and } P=0, \\ &\text{if } \gamma > 1, \quad \text{the sum converges, and } P>0. \end{aligned} \tag{9}$$

This means that when γ becomes greater than 1, there is a finite (nonzero) probability in the thermodynamic limit $N\to\infty$ that all particles go to the same state. This is an extreme case of clusterization, which is analogous to the Bose-Einstein condensation in a bosonic system, in which a finite fraction of a system's particles go to the same quantum state. The probability P that this happens goes continuously from 0 for $\gamma < 1$ to positive values for $\gamma > 1$. This behavior of P was verified numerically by a direct calculation of $P_N(N)$ from Eq. (6) for large N. The result is shown in Fig. 1(d). Using an integral approximation to the sum in Eq. (8), we can find how P scales in the vicinity of the transition point $\gamma_c=1$. Writing $\gamma=1+\epsilon$, for $\epsilon \ll 1$ we have

$$P \sim \exp(-1/\epsilon). \tag{10}$$

At $\epsilon=0$ ($\gamma=1$), P is nonanalytic, and displays an *essential singularity*, with all the derivatives vanishing as $\epsilon\to 0$.

Krapivsky *et al.* have examined the growth process of a complex network, where the attachment probability for a new node depends as a power law on the degree of the existing nodes, analogous to Eq. (1) [12]. They show that for $\gamma > 1$, a condensation phenomenon happens, where a node can receive a finite fraction of all the links of the network. This is somewhat similar to our results, even though our system has only two states, whereas the complex networks studied by Krapivsky *et al.* have many states, which can be identified with the network's nodes.

We now introduce a very simple model of gravitational aggregation to illustrate our theory, and to show that the "rich-get-richer" phenomenon can also happen in "hard" physical systems, and not only in the complex systems found in sociology, biology, etc. Suppose we have two bodies S_1 and S_2 whose positions are fixed in space, and have at some given time masses M_1 and M_2. Suppose further that they are far away from each other. What is the probability that a small incoming particle with mass m and energy E will be captured by S_1 (or S_2)? A rough approximation can be obtained by assuming that capture occurs if the particle enters the region where the gravitational potential energy due to S_1 is equal (in modulus) to the particle's kinetic energy E. The probability of capture by S_1 is then proportional to the cross-section area of the sphere given by $|U_1|=E$, where U_1 is the gravitational potential due to S_1. We have $U_1 \sim -M_1/R_1$, where R_1 is the distance from the center of body S_1, which is assumed to have spherical symmetry. Similar relations hold for S_2. Thus, for a given particle's energy E, the sphere has radius R_1 which is proportional to M_1, and since the capture probability P_1 is proportional to R_1^2, we find

$$P_i \sim M_i^2, \tag{11}$$

where P_i is the capture probability for S_i, $i=1,2$. When one of the bodies captures a particle, its mass increases, and from Eq. (11) its capture probability will increase, making it more likely to capture the next particle. If we take a sequence of incoming particles, we have a growth process like the one described by our theory. Assuming that all incoming particles have the same mass and the same energy, their masses M_i are proportional to the number n_i of captured particles. If we ignore particles that are not captured, we have exactly the growth model described by Eq. (1), with $\gamma=2$. We conclude that this is a statistically unpredictable system, where with overwhelming probability one of the bodies will end up with nearly the entirety of the mass.

The fact that the previous system is statistically unpredictable is due to the presence of a long-range interaction, such as gravity. To see this, consider an aggregation process without gravity. In this case, the capture probability P_i is proportional to the cross-section area of the bodies S_1 and S_2 themselves. Assuming S_1 and S_2 have constant densities, we find $P_i \sim M_i^{2/3}$. We still have a clustering tendency, but in this case $\gamma < 1$. This means that the aggregation process will yield approximately equal masses to S_1 and S_2. We think this is a very general result: if in an aggregation process there is a long-range force, it is statistically unpredictable, whereas if only short-range interactions are present, it is statistically predictable, and the usual statistical properties hold.

To conclude, we notice that, although we focused on the case of two states throughout the paper, the above theory can be easily generalized for any finite number m of states. In this case, the generalization of Eq. (1) is

$$p_i(n_1, \cdots, n_m) = \frac{n_i^\gamma}{n_1^\gamma + \cdots + n_m^\gamma}, \tag{12}$$

where $p_i(n_1,\cdots,n_m)$ is the probability that the next particle will go to state S_i, with $i=1,2,\cdots,m$. The master equation for m states is an obvious extension of Eq. (2), and the same mean-field approximation can be made, with a result similar to Eq. (5):

$$\begin{aligned} &\text{if } r_{ij}^{\gamma-1} > 1, \quad \dot{r}_{ij} > 0, \\ &\text{if } r_{ij}^{\gamma-1} < 1, \quad \dot{r}_{ij} < 0, \end{aligned} \tag{13}$$

where $r_{ij}=n_i/n_j$, $i\neq j$, and $i,j=1,\cdots,m$. We thus arrive at a conclusion similar to the $m=2$ case: for $\gamma<1$, the ratios r_{ij} go to 1 for $N=\Sigma_i n_i\to\infty$, and the occupation of all states will be nearly equal, $n_1\approx n_2\approx\cdots\approx n_m\approx M/m$. For $\gamma>1$, on the other hand, one state will be occupied with the overwhelming majority of the particles, whereas the other states will be practically empty. In particular, the gravitational aggregation by m bodies will have the same clustering properties as in the case of 2 bodies.

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[1] Y. Ijiri and H. A. Simon, *Skew Distributions and the Size of Business Firms* (Elsevier–North Holland, New York, 1977).
[2] G. K. Zipf, *Human Behavior and the Principle of Least Effort* (Addison-Wesley, Cambridge, MA, 1949).
[3] M. Levy and S. Solomon, Int. J. Mod. Phys. C **7**, 595 (1996).
[4] R. K. Merton, Science **159**, 56 (1968).
[5] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[6] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002); M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[7] M. Faloutsos *et al.*, Comput. Commun. Rev. **29**, 251 (1999).
[8] R. Albert *et al.*, Nature (London) **401**, 130 (1999).
[9] J. M. Montoya and R. V. Sole, J. Theor. Biol. **214**, 405 (2002); H. Jeong *et al.*, Nature (London) **407**, 651 (2000).
[10] A. E. Motter *et al.*, Phys. Rev. E **65**, 065102(R) (2002).
[11] W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1964).
[12] P. L. Krapivsky, S. Redner, and F. Leyvraz, Phys. Rev. Lett. **85**, 4629 (2000); P. L. Krapivsky and S. Redner, Phys. Rev. E **63**, 066123 (2001).
[13] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, UK, 1999).
[14] M. N. Barber and B. W. Ninham, *Random and Restricted Walks* (Gordon and Breach, New York, 1970).
[15] R. Cafiero, A. Gabrielli, and M. A. Muñoz, Phys. Rev. E **57**, 5060 (1998).